

Rational Combinatorics

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Abstract

We propose a categorical setting for the study of the combinatorics of rational numbers. We find combinatorial interpretation for Bernoulli and Euler numbers and polynomials.

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1 Introduction

Let Cat be the category whose objects are small categories (categories whose collection of objects are sets) and whose morphisms $Cat(\mathcal{C}, \mathcal{D})$ from category \mathcal{C} to category \mathcal{D} are functors $F : \mathcal{C} \rightarrow \mathcal{D}$. Let Set be the category of sets. We define an equivalence relation $Isoc$ on $Ob(\mathcal{C})$, objects of \mathcal{C} , as follows: x and y are equivalent if and only if there exists an isomorphism $f \in \mathcal{C}(x, y)$. There is a natural functor $D : Cat \rightarrow Set$ called *deategorification* given by $D(\mathcal{C}) = Ob(\mathcal{C})/Isoc$ for any small category \mathcal{C} . Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ then $D(F) : Ob(\mathcal{C})/Isoc \rightarrow Ob(\mathcal{D})/Isoc$ is the induced map. If x is a set and \mathcal{C} is a category such that $D(\mathcal{C}) = x$, we say that \mathcal{C} is a *categorification* of x . The reader may consult [1] and [5] for more on the notion of categorification.

We are interested in the categorification of sets with additional properties, for example one would like to find out what is the categorification of a ring. It turns out that the definition of categorification given above is too rigid, for most applications a weaker notion seems to be more useful. For example, see Section 2 for details, a categorification of a ring R is a category \mathcal{C} together with bifunctors \oplus and \otimes from $\mathcal{C} \times \mathcal{C}$ to \mathcal{C} , distinguished objects 0 and 1 , and a negative functor N from \mathcal{C} to \mathcal{C} . Moreover \mathcal{C} should be provided with a valuation map $|\cdot| : Ob(\mathcal{C}) \rightarrow R$ such that $|a| = |b|$ if a and b are isomorphic, $|a \oplus b| = |a| \oplus |b|$, $|a \otimes b| = |a||b|$, $|1| = 1$ and $|0| = 0$, and $|N(a)| = -|a|$, for any objects a and b in \mathcal{C} . Categorification of semi-rings is defined similarly but the functor N is no longer required.

Our starting point is the theory of combinatorial species of Joyal, see [7] and [8], which can be described as starting from a categorification of the natural numbers \mathbb{N} and extending it to a categorification of the semi-ring $\mathbb{N}[[x]]$ of formal power series with coefficients in \mathbb{N} . Namely, the category *set* of finite sets with disjoint union and Cartesian product, together with the map $|\cdot| : Ob(set) \rightarrow \mathbb{N}$ sending a finite set x to its cardinality $|x|$, is a categorification of the natural numbers. Joyal goes on and shows that the category $set^{\mathbb{B}}$ of functors from \mathbb{B} to *set* defines a

categorification of the semi-ring $\mathbb{N}[[x]]$.

As explained by Zeilberger in [14] the main topic of enumerative natural combinatorics is the following: given a infinite sequence $x_0, x_1, \dots, x_n, \dots$ of finite sets, objects of *set*, compute the corresponding sequence $|x_0|, |x_1|, \dots, |x_n|, \dots$ of natural numbers. So one looks for a numerical representation of combinatorial objects. There is also an inverse problem in enumerative combinatorics: given a sequence of natural numbers $a_0, a_1, \dots, a_n, \dots$ find an appropriated sequence of finite sets $x_0, x_1, \dots, x_n, \dots$ such that $|x_i| = a_i$ for $i \in \mathbb{N}$. Here we look for a combinatorial interpretation of a sequence of natural numbers. In the theory of species a fundamental part is played by the groupoid \mathbb{B} whose objects are finite sets and whose morphisms are bijections. From the point of view of the theory of species the main problem of enumerative natural combinatorics can be described as follows: given a functor $F : \mathbb{B} \rightarrow \text{set}$ find the sequence $|F([0])|, |F([1])|, \dots, |F([n])|, \dots$ or what is essentially the same, find the associated generating series $\sum_{n=0}^{\infty} |F([n])| \frac{x^n}{n!}$.

Enumerative combinatorics can be extended to deal with integer numbers. The main problem of enumerative integral combinatorics is the following: given a sequence $(x_0, y_0), \dots, (x_n, y_n), \dots$ of pairs of finite sets, objects of $\mathbb{Z}_2\text{-set}$ to be defined in Section 4, compute the corresponding sequence $|(x_0, y_0)|, \dots, |(x_n, y_n)|, \dots$ of integers, where for each pair of finite sets (x, y) its cardinality is defined by $|(x, y)| = |x| - |y|$. The inverse problem is the following: given a sequence a_0, \dots, a_n, \dots of integers find a nice sequence $(x_0, y_0), \dots, (x_n, y_n), \dots$ of pairs of finite sets such that $|(x_i, y_i)| := |x_i| - |y_i| = a_i$ for $i \in \mathbb{N}$. From the species point of view the main problem of integral combinatorics may be described as follows: given a functor $F : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-set}$ compute the associated generating series $\sum_{n=0}^{\infty} |F([n])| \frac{x^n}{n!}$.

We face the following problem in this paper: find a categorification of \mathbb{Q} the ring of rational numbers. In the first two sections of this paper we concentrate on the problem of the categorification of $\mathbb{Q}_{\geq 0}$, the semi-ring of nonnegative rational numbers, leaving the study of the categorification of \mathbb{Q} for the final two sections. So we need a category \mathcal{C} provided with sum and product bifunctors together with a valuation map $|\cdot| : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{Q}_{\geq 0}$ satisfying a natural set of axioms. Given such pair \mathcal{C} and $|\cdot|$ one defines the main problem of "rational combinatorics" as the problem of finding the sequence $|x_0|, \dots, |x_n|, \dots$ for any sequence x_0, \dots, x_n, \dots of objects of \mathcal{C} . Similarly the inverse problem in "rational combinatorics" would be the following: given a sequence a_0, \dots, a_n, \dots in $\mathbb{Q}_{\geq 0}$, find a nice sequence x_0, \dots, x_n, \dots of objects of \mathcal{C} such that $|x_i| = a_i$, for $i \in \mathbb{N}$. There are several categories \mathcal{C} provided with a valuation map $|\cdot| : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{Q}_{\geq 0}$, so if we like our enumerative problem above to be consider as being "combinatorial" we should demand that the category \mathcal{C} be close to *set*, and the valuation map $|\cdot|$ close to the notion of cardinality of finite sets.

Fortunately, Baez and Dolan in [2] have proposed a good candidate to play the part of *set* when dealing with the combinatorial properties of rational numbers, namely the category *gpd* of finite groupoids. A groupoid is a category such that all its morphisms are invertible. According to Baez and Dolan the cardinality of a finite groupoid G is given by the formula $|G| = \sum_{x \in D(G)} \frac{1}{|G(x, x)|}$. Starting from this definition we construct a categorification of the ring \mathbb{Q} of rational numbers. This construction can be further generalized using Joyal's theory of

species to yield a categorification of the ring $\mathbb{Q}[[x_1, \dots, x_n]]$ of formal power series with rational coefficients in n -variables. This paper is devoted to highlight the properties that make *gpd* into a good ground set for the study of the combinatorial properties of rational numbers, however we mention from the outset its main shortcoming: it does not seem to exist a functorial way to associate to each groupoid G another groupoid G^{-1} such that $|G^{-1}| = |G|^{-1}$.

2 Categorification of rings

In this section we introduce the notion of categorification of rings. A *categorification* of a ring R is a triple $(\mathcal{C}, N, |\cdot|)$ such that \mathcal{C} is a category, $N : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $|\cdot| : \text{Ob}(\mathcal{C}) \rightarrow R$ is a map called the valuation map. This data is required to satisfy the following axioms:

1. \mathcal{C} is provided with bifunctors $\oplus : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called sum and product, respectively. Functors \oplus and \otimes are such that
 - There are distinguished objects 0 and 1 in \mathcal{C} .
 - The triple $(\mathcal{C}, \oplus, 0)$ is a symmetric monoidal category with unit 0.
 - The triple $(\mathcal{C}, \otimes, 1)$ is a monoidal category with unit 1.
 - Distributivity holds. That is for objects a, b, c of \mathcal{C} there are natural isomorphisms $a \otimes (b \oplus c) \simeq (a \otimes b) \oplus (a \otimes c)$ and $(a \oplus b) \otimes c \simeq (a \otimes c) \oplus (b \otimes c)$.
2. The functor $N : \mathcal{C} \rightarrow \mathcal{C}$ must be such that for objects a, b of \mathcal{C} the following identities hold
 - $N(a \oplus b) = N(a) \oplus N(b)$.
 - $N(0) = 0$.
 - $N^2 = I$ (identity functor).
3. The map $|\cdot| : \text{Ob}(\mathcal{C}) \rightarrow R$ is such that for objects a, b of \mathcal{C} the following identities hold
 - $|a| = |b|$ if a and b are isomorphic.
 - $|a \oplus b| = |a| + |b|$.
 - $|a \otimes b| = |a||b|$.
 - $|1| = 1$ and $|0| = 0$.
 - $|N(a)| = -|a|$.

Let us make a few remarks regarding the notion of categorification of rings.

1. If R is a semi-ring, i.e., we do not assume the existence of additive inverses in R , then a categorification of R is defined as above but we omit the existence of the functor N .
2. N is called the negative functor. In practice we prefer to write $-a$ instead of $N(a)$.
3. Notice that we are not requiring that \oplus and \otimes be the coproduct and product of \mathcal{C} , although in several examples they do agree.

4. See Laplaza's works [10] and [11] for the full definition, and coherence theorems, of a category with two monoidal structures satisfying the distributive property.
5. We stress that we only demand that $|a \oplus N(a)| = 0$. Demanding the stronger condition $a \oplus N(a)$ isomorphic to 0, would reduce drastically the scope of our definition.
6. We call $|a|$ the valuation of a . A categorification is surjective if the valuation map is surjective.
7. R is a categorification of itself if we consider R as the category whose object set is R , and identities as morphisms. The valuation map is the identity map, and the negative $N(r)$ of $r \in R$ is $-r$. Thus, there is not existence problem related with the notion of categorification: all rings admit a categorification. The philosophy behind the notion of categorification is that we can obtain valuable information about a ring R by looking at its various categorifications.

We close this section giving a couple of examples of categorifications. Consider the category set whose objects are finite sets and whose morphisms are maps. The following data describes set as a surjective categorification of \mathbb{N} .

- The triple (set, \sqcup, \emptyset) , where $\sqcup : set \times set \rightarrow set$ is disjoint union, is a symmetric monoidal category with unit \emptyset .
- The triple $(set, \times, \{1\})$, where $\times : set \times set \rightarrow set$ is Cartesian product, is a monoidal category with unit $\{1\}$.
- The valuation map $| \cdot | : Ob(set) \rightarrow \mathbb{N}$ sends a finite set x to its cardinality $|x|$.

Let set^n be the category $set \times \dots \times set$. Objects in set^n are pairs (x, f) where x is a finite set and $f : x \rightarrow \{1, \dots, n\}$ is a map. Morphisms in set^n from (x, f) to (y, g) are maps $\alpha : x \rightarrow y$ such that $g \circ \alpha = f$. Alternatively, objects of set^n can be describe as n -tuples (x_1, \dots, x_n) of finite sets. A morphism in set^n from (x_1, \dots, x_n) to (y_1, \dots, y_n) is given by a n -tuple of maps $(\alpha_1, \dots, \alpha_n)$ such that $\alpha_i : x_i \rightarrow y_i$ for $1 \leq i \leq n$. The following data describes set^n as a surjective categorification of \mathbb{N}^n .

- The triple $(set^n, \sqcup, \emptyset)$ is a symmetric monoidal category with unit $\emptyset = (\emptyset, \dots, \emptyset)$. Disjoint union $\sqcup : set^n \times set^n \rightarrow set^n$ is given by

$$(x_1, \dots, x_n) \sqcup (y_1, \dots, y_n) = (x_1 \sqcup y_1, \dots, x_n \sqcup y_n).$$

- The triple $(set^n, \times, \{1\})$ is a monoidal category with unit $\{1\} = (\{1\}, \dots, \{1\})$. Cartesian product $\times : set^n \times set^n \rightarrow set^n$ is given by

$$(x_1, \dots, x_n) \times (y_1, \dots, y_n) = (x_1 \times y_1, \dots, x_n \times y_n).$$

- The valuation map $| \cdot | : Ob(set^n) \rightarrow \mathbb{N}^n$ sends (x_1, \dots, x_n) into $(|x_1|, \dots, |x_n|)$.

In the construction above we have implicitly made use of the fact that if \mathcal{C}_1 and \mathcal{C}_2 are categorifications of rings R_1 and R_2 , respectively, then $\mathcal{C}_1 \times \mathcal{C}_2$ is a categorification of $R_1 \times R_2$, with sum, product, negative functor and valuation map defined componentwise.

3 Nonnegative rational species

We begin this section introducing groupoids and their cardinality, then we provide a list of useful examples of groupoids. We use the following notations $[n] = \{1, 2, \dots, n\}$, $S_n = \{f : [n] \rightarrow [n] \mid f \text{ is bijective}\}$ is the symmetric group on n letters, $\mathbb{Z}_n = \{0, \dots, n-1\}$ is the cyclic group of order n .

Definition 1. A groupoid G is a category such that all its morphisms are invertible. We denote by Gpd the full subcategory of Cat whose objects are groupoids.

Let us introduce a few examples of groupoids.

Example 2. Any category \mathcal{C} has an underlying groupoid, which has the same objects as \mathcal{C} and whose morphisms are isomorphisms in \mathcal{C} .

Example 3. \mathbb{B} denotes the groupoid whose objects are finite sets and whose morphisms are bijections between finite sets. \mathbb{B} is the underlying groupoid of Set .

Example 4. \mathbb{B}^n denotes the groupoid whose objects are pairs (x, f) , where x is a finite set and $f : x \rightarrow [n]$ is a map. Morphisms in \mathbb{B}^n from (x, f) to (y, g) are bijections $\alpha : x \rightarrow y$ such that $g \circ \alpha = f$. \mathbb{B}^n is the underlying groupoid of set^n .

Example 5. A group G may be regarded as the groupoid \overline{G} with one object 1 and $\overline{G}(1, 1) = G$.

Definition 6. A groupoid G is called finite if its set of objects is finite, and $G(x, y)$ is a finite set for x, y objects of G . We denote by gpd the full subcategory of Gpd whose objects are finite groupoids.

Disjoint union and Cartesian product are given on gpd as restrictions of the corresponding functors on Cat , see [12]. The disjoint union $\mathcal{C} \sqcup \mathcal{D}$, of categories \mathcal{C} and \mathcal{D} , is the category with objects $Ob(\mathcal{C}) \sqcup Ob(\mathcal{D})$, and morphisms from x to y given by

$$\mathcal{C} \sqcup \mathcal{D}(x, y) = \begin{cases} \mathcal{C}(x, y) & \text{if } x, y \in Ob(\mathcal{C}), \\ \mathcal{D}(x, y) & \text{if } x, y \in Ob(\mathcal{D}), \\ \emptyset & \text{otherwise.} \end{cases}$$

The Cartesian product $\mathcal{C} \times \mathcal{D}$, of categories \mathcal{C} and \mathcal{D} , is the category such that

$$Ob(\mathcal{C} \times \mathcal{D}) = Ob(\mathcal{C}) \times Ob(\mathcal{D}),$$

and for $(x_1, y_1), (x_2, y_2) \in Ob(\mathcal{C} \times \mathcal{D})$ we have $\mathcal{C} \times \mathcal{D}((x_1, y_1), (x_2, y_2)) = \mathcal{C}(x_1, x_2) \times \mathcal{D}(y_1, y_2)$. The unit for disjoint union is the category \emptyset with no objects. A unit for Cartesian product is the category 1 with one object and one morphism.

Let $\mathbb{Q}_{\geq 0}$ be the semi-ring of nonnegative rational numbers. Recall that $D(G)$ denotes the set of isomorphisms classes of objects in the groupoid G .

Theorem 7. The map $| \cdot | : Ob(gpd) \rightarrow \mathbb{Q}_{\geq 0}$ given by $|G| = \sum_{x \in D(G)} \frac{1}{|G(x, x)|}$ is a surjective $\mathbb{Q}_{\geq 0}$ -valuation on gpd .

Proof. Let G and H be finite groupoids, then

$$\begin{aligned} |G \sqcup H| &= \sum_{x \in D(G \sqcup H)} \frac{1}{|(G \sqcup H)(x, x)|} = \sum_{x \in D(G)} \frac{1}{|G(x, x)|} + \sum_{x \in D(H)} \frac{1}{|H(x, x)|} \\ &= |G| + |H|, \end{aligned}$$

and

$$\begin{aligned} |G \times H| &= \sum_{(x, y) \in D(G \times H)} \frac{1}{|G \times H(x, y)|} = \sum_{(x, y) \in D(G) \times D(H)} \frac{1}{|G(x, x)| |H(y, y)|} \\ &= \left(\sum_{x \in D(G)} \frac{1}{|G(x, x)|} \right) \left(\sum_{y \in D(H)} \frac{1}{|H(y, y)|} \right) = |G| |H|. \end{aligned}$$

$|1| = 1$ and $|\emptyset| = 0$. For each $\frac{a}{b} \in \mathbb{Q}_{\geq 0}$ the groupoid $\mathbb{Z}_b^{\sqcup a}$ satisfies $|\mathbb{Z}_b^{\sqcup a}| = \frac{a}{b}$. \square

Following Baez and Dolan we call $|G|$ the cardinality of the groupoid G . The outcome of Theorem 7 is that the pair $(gpd, |\cdot|)$ is a surjective categorification of the semi-ring $\mathbb{Q}_{\geq 0}$. Notice that any finite set x may be regarded as the finite groupoid whose set of objects is x and with identity morphisms only. We have an inclusion $i : set \rightarrow gpd$ such that $|x| = |i(x)|$.

Given categories \mathcal{C} and \mathcal{D} we let $\mathcal{C}^{\mathcal{D}}$ be the category whose objects are functors $F : \mathcal{D} \rightarrow \mathcal{C}$. Morphisms from F to G in $\mathcal{C}^{\mathcal{D}}$ are natural transformations $T : F \rightarrow G$.

Definition 8. *The category of \mathbb{B} -gpd species is the category $gpd^{\mathbb{B}}$. An object $F : \mathbb{B} \rightarrow gpd$ is called a \mathbb{B} -gpd species or a (nonnegative) rational species. The category of (nonnegative) rational species in n -variables is $gpd^{\mathbb{B}^n}$.*

We denote by $gpd_0^{\mathbb{B}^n}$ the full subcategory of $gpd^{\mathbb{B}^n}$ whose objects are functors $F : \mathbb{B}^n \rightarrow gpd$ such that $F(\emptyset) = \emptyset$. Next we define operations on rational species, which indeed apply as well to $Gdp^{\mathbb{B}^n}$. These operations may be regarded as examples of a fairly general construction given in [5]. We let $\Pi[x]$ be the set of partitions of x .

Definition 9. *Let $F, G \in Ob(gpd^{\mathbb{B}^n})$, $G_1, \dots, G_n \in Ob(gpd_0^{\mathbb{B}^n})$ and (x, f) be an object of \mathbb{B}^n . The following formulae define product, composition and derivative of \mathbb{B}^n -gpd species*

1. $(F + G)(x, f) = F(x, f) \sqcup G(x, f)$.
2. $(FG)(x, f) = \bigsqcup_{a \sqcup b = x} F(a, f|_a) \times G(b, f|_b)$.
3. $(F \times G)(x, f) = F(x, f) \times G(x, f)$.
4. $F(G_1, \dots, G_n)(x, f) = \bigsqcup_{\substack{\pi \in \Pi[x] \\ g : \pi \rightarrow [n]}} F(\pi, g) \times \prod_{b \in \pi} G_{g(b)}(b, f|_b)$.
5. For $i \in [n]$, $\partial_i : gpd^{\mathbb{B}^n} \rightarrow gpd^{\mathbb{B}^n}$ is the functor such that $\partial_i(F)$ is given by

$$\partial_i F(x, f) = F(x \sqcup \{*\}, f \sqcup \{(*, i)\}).$$

For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ we write $[a] = ([a_1], \dots, [a_n])$, $a! = a_1! \dots a_n!$ and $x^a = x_1^{a_1} \dots x_n^{a_n}$. The isomorphism class of an object (x, f) in \mathbb{B}^n is given by the n -tuple $(|f^{-1}(i)|)_{i=1}^n \in \mathbb{N}^n$.

Theorem 10. *The map $|\cdot| : \text{Ob}(\text{gpd}^{\mathbb{B}^n}) \longrightarrow \mathbb{Q}_{\geq 0}[[x_1, \dots, x_n]]$ given by*

$$|F|(x_1, \dots, x_n) = \sum_{a \in \mathbb{N}^n} |F([a])| \frac{x^a}{a!}$$

is a $\mathbb{Q}_{\geq 0}[[x_1, \dots, x_n]]$ -valuation on $\text{gpd}^{\mathbb{B}^n}$. Moreover $|F \times G| = |F| \times |G|$, $|\partial_i F| = \partial_i |F|$ and

$$|F(G_1, \dots, G_n)| = |F|(|G_1|, \dots, |G_n|).$$

Above \times is the Hadamard product of series and G_i is assumed to be in $\text{gpd}_0^{\mathbb{B}^n}$ for $1 \leq i \leq n$.

The valuation $|F|$ of a rational species F is called its generating series. The outcome of Theorem 10 is that $(\text{gpd}^{\mathbb{B}^n}, |\cdot|)$ is a surjective categorification of the semi-ring $\mathbb{Q}_{\geq 0}[[x_1, \dots, x_n]]$. We proceed to introduce a list of examples of rational species computing in each case the corresponding generating series.

Example 11. *For $1 \leq i \leq n$ the singular species $X_i : \mathbb{B}^n \rightarrow \text{gpd}$ is such that*

$$X_i(x, f) = \begin{cases} 1 & \text{if } |x| = 1 \text{ and } f(x) = i, \\ \emptyset & \text{otherwise.} \end{cases}$$

for (x, f) in \mathbb{B}^n . Clearly $|X_i| = x_i \in \mathbb{Q}_{\geq 0}[[x_1, \dots, x_n]]$.

Example 12. *Let $S^N : \mathbb{B} \rightarrow \text{gpd}$ be the species sending a finite set x to the groupoid $S^N(x)$ given by $\text{Ob}(S^N(x)) = \{x\}$ and $S^N(x)(x, x) = S_{|x|}^N$. We have*

$$|S^N| = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^{N+1}}.$$

Example 13. *Let $\mathbb{Z}^N : \mathbb{B} \rightarrow \text{gpd}$ be such that for a finite set x the groupoid $\mathbb{Z}^N(x)$ is given by $\text{Ob}(\mathbb{Z}^N(x)) = \begin{cases} \{x\} & \text{if } x \neq \emptyset \\ \emptyset & \text{if } x = \emptyset \end{cases}$ and $\mathbb{Z}^N(x)(x, x) = \mathbb{Z}_{|x|}^N$ for x nonempty. We have*

$$|\mathbb{Z}^N| = \sum_{n=1}^{\infty} \frac{1}{n^N} \frac{x^n}{n!}.$$

For $N = 1$ we obtain

$$|\mathbb{Z}| = \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{n!} = \int \frac{e^x - 1}{x} dx.$$

Example 14. *Let $E_N : \mathbb{B} \rightarrow \text{gpd}$ be such that for a finite set x the groupoid $E_N(x)$ is given by $\text{Ob}(E_N(x)) = \{x\}$ and $E_N(x)(x, x) = \mathbb{Z}_{|x|}^N$. We have*

$$|E_N| = \sum_{n=0}^{\infty} \frac{1}{N^n} \frac{x^n}{n!} = e^{\frac{x}{N}}.$$

Example 15. Let G be a group. Let $\overline{G} : \mathbb{B} \rightarrow \text{gpd}$ be the rational species such that $\overline{G}(\emptyset) = \overline{G}$ and $\overline{G}(x) = \emptyset$ if x is nonempty. Clearly $|\overline{G}| = \frac{1}{|G|}$.

Example 16. Let $(\mathbb{Z}_N)^{(\cdot)} : \mathbb{B} \rightarrow \text{gpd}$ be the rational species such that for a finite set x the groupoid $(\mathbb{Z}_N)^{(x)}$ is given by

$$\text{Ob}\left((\mathbb{Z}_N)^{(x)}\right) = \begin{cases} \{x\} & \text{if } x \neq \emptyset \\ \emptyset & \text{if } x = \emptyset, \end{cases}$$

and

$$(\mathbb{Z}_N)^{(x)}(x, x) = \mathbb{Z}_N \times \mathbb{Z}_{N+1} \times \dots \times \mathbb{Z}_{N+|x|-1}.$$

We have

$$\left|(\mathbb{Z}_N)^{(\cdot)}\right| = \sum_{n=1}^{\infty} \frac{1}{(N)^{(n)}} \frac{x^n}{n!}.$$

where $(a)^{(b)} = a(a+1) \dots (a+b-1)$ is the increasing factorial [13].

Example 17. Let $P : \mathbb{B} \rightarrow \text{gpd}$ be the rational species such that $\text{Ob}(P(x)) = \{a \mid a \subseteq x\}$. Morphisms are given by $P(x)(a, b) = \{\alpha : a \rightarrow b \mid \alpha \text{ is bijective}\}$. We have

$$|P| = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{1}{k!} \right) \frac{x^n}{n!}$$

If f and g are formal power series such that $\frac{d^N g(x)}{dx^N} = f(x)$, then we write $g(x) = \int^{(N)} f(x) dx$.

Definition 18. The increasing factorial rational species $\mathbb{Z}^{(N)} : \mathbb{B} \rightarrow \text{gpd}$ is such that for each finite set x the groupoid $\mathbb{Z}^{(N)}(x)$ is given by

$$\text{Ob}(\mathbb{Z}^{(N)}(x)) = \begin{cases} \{x\} & x \neq \emptyset \\ \emptyset & \text{if } x = \emptyset, \end{cases}$$

and $\mathbb{Z}^{(N)}(x)(x, x) = \mathbb{Z}_{|x|} \times \mathbb{Z}_{|x|+1} \times \dots \times \mathbb{Z}_{|x|+N-1}$ for x nonempty.

Theorem 19.

$$\left|\mathbb{Z}^{(N)}\right| = x^{1-N} \int^{(N)} \frac{e^x - 1}{x} dx.$$

Proof.

$$\begin{aligned} \left|\mathbb{Z}^{(N)}\right| &= \sum_{n=1}^{\infty} \frac{1}{(n)^{(N)}} \frac{x^n}{n!} = \frac{1}{x^{N-1}} \sum_{n=1}^{\infty} \frac{1}{n} \int^{(N-1)} \frac{x^n}{n!} dx = \frac{1}{x^{N-1}} \int^{(N-1)} \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{n!} \right) dx \\ &= \frac{1}{x^{N-1}} \int^{(N-1)} \left(\int \frac{e^x - 1}{x} dx \right) dx = x^{1-N} \int^{(N)} \frac{e^x - 1}{x} dx. \end{aligned}$$

□

Let G be a group acting on a finite set x . The quotient groupoid x/G is such that $Ob(x/G) = x$ and $x/G(a, b) = \{g \in G \mid ga = b\}$ for $a, b \in x$. The decategorification $D(x/G)$ of x/G is just the quotient set of the action of G on x . Below we use the notation $O(a) = \{ga \mid g \in G\}$.

Proposition 20. *If G acts on x then $|x/G| = \frac{|x|}{|G|}$.*

Proof.

$$|x/G| = \sum_{\bar{a} \in D(x/G)} \frac{1}{|\text{Iso}(a)|} = \sum_{\bar{a} \in D(x/G)} \frac{|O(a)|}{|G|} = \frac{1}{|G|} \sum_{\bar{a} \in D(x/G)} |O(a)| = \frac{|x|}{|G|}.$$

□

Let G be a subgroup of S_k .

Example 21. Let $P_G : \mathbb{B} \rightarrow \text{gpd}$ be such that for a finite set x the groupoid $P_G(x)$ is the quotient groupoid x^k/G . It should be clear that $|P_{S_k}| = \sum_{n=1}^{\infty} \frac{n^k}{k!} \frac{x^n}{n!}$ and $|P_{\mathbb{Z}_k}| = \sum_{n=1}^{\infty} \frac{n^k}{k} \frac{x^n}{n!}$.

The inertia functor $I : \text{gpd} \rightarrow \text{gpd}$ is such that for each G the groupoid $I(G)$ is given by

$$Ob(I(G)) = Ob(G),$$

and for objects a and b of $I(G)$ we set

$$I(G)(a, b) = \begin{cases} G(a, a) & \text{if } a = b, \\ \emptyset & \text{otherwise.} \end{cases}$$

For example the set of objects of $I(x^k/S_k)$ is x^k . For $a, b \in x^k$ we have

$$I(x^k/S_k)(a, b) = \begin{cases} \{\sigma \in S_k \mid a\sigma = a\} & \text{if } a = b, \\ \emptyset & a \neq b. \end{cases}$$

One checks that

$$|\{\sigma \in S_k \mid a\sigma = a\}| = \prod_{i \in x} |a^{-1}(i)|!.$$

Therefore if we assume that $|x| = n$ we get

$$\left| I(x^k/S_k) \right| = \sum_{a: [k] \rightarrow x} \frac{1}{\prod_{i \in X} |a^{-1}(i)|!} = \sum_{s_1 + \dots + s_n = k} \binom{k}{s_1, \dots, s_n} \frac{1}{s_1! \dots s_n!}.$$

We extend the inertia functor to rational species $I : \text{gpd}^{\mathbb{B}} \rightarrow \text{gpd}^{\mathbb{B}}$ by the rule $I(F)(x) = I(F(x))$ for any finite set x . With this notation we have shown the following result.

Proposition 22.

$$|I(P_{S_k})| = \sum_{n=0}^{\infty} \left(\sum_{s_1 + \dots + s_n = k} \frac{k}{(s_1! \dots s_n!)^2} \right) \frac{x^n}{n!}.$$

Let us introduce other interesting examples of rational species. Let $\text{Isinh} : \mathbb{B} \rightarrow \text{gpd}$ be such that for a finite set x the groupoid $\text{Isinh}(x)$ is given by

$$\text{Ob}(\text{Isinh}(x)) = \begin{cases} \{x\} & \text{if } |x| \text{ is odd,} \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\text{Isinh}(x)(x, x) = \mathbb{Z}_{|x|} \text{ for } |x| \text{ odd.}$$

Proposition 23.

$$|\text{Isinh}| = \int \frac{\sinh(x)}{x} dx.$$

Proof.

$$|\text{Isinh}| = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \int \frac{\sinh(x)}{x} dx.$$

□

The rational species $\text{Icosh} : \mathbb{B} \rightarrow \text{gpd}$ is such that for a finite set x the groupoid $\text{Icosh}(x)$ is given by

$$\text{Ob}(\text{Icosh}(x)) = \begin{cases} \{x\} & \text{if } |x| \text{ is even,} \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\text{Icosh}(x)(x, x) = \mathbb{Z}_{|x|} \text{ for } x \text{ even.}$$

Proposition 24.

$$|\text{Icosh}| = \int \frac{\cosh(x)}{x} dx.$$

Proof.

$$|\text{Icosh}| = \sum_{n=0}^{\infty} \frac{1}{2n} \frac{x^{2n}}{(2n)!} = \int \frac{\cosh(x)}{x} dx.$$

□

Let us closed this section by introducing a couple of species that will be used below.

Example 25. The species $1 \in \text{gpd}^{\mathbb{B}^n}$ is such that for each object (x, f) of \mathbb{B}^n

$$1(x, f) = \begin{cases} 1 & \text{if } x = \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly $|1| = 1$.

Example 26. The exponential species $\text{Exp} \in \text{gpd}^{\mathbb{B}^n}$ is given by $\text{Exp}(x, f) = 1$. Clearly we have $|\text{Exp}| = e^{x_1 + \dots + x_n}$.

4 Rational species

So far we have consider only the categorification of semi-rings. In order to proceed further and, in particular, make sense of multiplicative inverses in the category of species it becomes necessary to deal with negative species. As explained in Section 2 the categorification of a ring requires in addition to the existence of sum, product, units and valuation, the existence of a negative functor. In this section we are going to construct categorifications of the rings \mathbb{Z} , $\mathbb{Z}[[x_1, \dots, x_n]]$, \mathbb{Q} and $\mathbb{Q}[[x_1, \dots, x_n]]$, but we stress that our techniques may be applied to many other rings as well.

We begin studying the categorification of \mathbb{Z} . Consider the category $\mathbb{Z}_2\text{-set} = \text{set} \times \text{set}$. Objects in $\mathbb{Z}_2\text{-set}$ are pairs of finite sets. Morphisms from (a_1, b_1) to (a_2, b_2) , objects of $\mathbb{Z}_2\text{-set}$, are given by

$$\mathbb{Z}_2\text{-set}((a_1, b_1), (a_2, b_2)) = \text{set}(a_1, a_2) \times \text{set}(b_1, b_2).$$

The disjoint union bifunctor $\sqcup : \mathbb{Z}_2\text{-set} \times \mathbb{Z}_2\text{-set} \rightarrow \mathbb{Z}_2\text{-set}$ is given by

$$(a_1, b_1) \sqcup (a_2, b_2) = (a_1 \sqcup a_2, b_1 \sqcup b_2).$$

The Cartesian product bifunctor is given by

$$(a_1, b_1) \times (a_2, b_2) = (a_1 \times a_2 \sqcup b_1 \times b_2, a_1 \times b_2 \sqcup a_2 \times b_1).$$

The distinguished objects \emptyset and 1 are $\emptyset = (\emptyset, \emptyset)$ and $1 = (1, \emptyset)$, respectively. The valuation map $|\cdot| : \mathbb{Z}_2\text{-set} \rightarrow \mathbb{Z}$ is given by $|(a, b)| = |a| - |b|$, the negative functor $N : \mathbb{Z}_2\text{-set} \rightarrow \mathbb{Z}_2\text{-set}$ is given by $N(a, b) = (b, a)$ for (a, b) in $\mathbb{Z}_2\text{-set}$. With these definitions we obtain the following result.

Theorem 27. *$(\mathbb{Z}_2\text{-set}, N, |\cdot|)$ is a surjective categorification of \mathbb{Z} .*

The categorifications set of \mathbb{N} and $\mathbb{Z}_2\text{-set}$ of \mathbb{Z} are fundamentally different. In set objects x and y are isomorphic if and only if $|x| = |y|$. That is not the case in $\mathbb{Z}_2\text{-set}$, where for example $|([2], [4])| = |([11], [13])| = -2$ but $([2], [4])$ and $([11], [13])$ are not isomorphic. At first sight this may seem like a nuance, however, there is nothing wrong with the fact that $([2], [4])$ and $([11], [13])$ are not isomorphic, since it is rather intuitive that they provide different combinatorial interpretations for -2 . Indeed the pair $([2], [4])$ leads to an interpretation of -2 as a difference of even numbers, while the pair $([11], [13])$ leads to an interpretation of -2 as a difference of prime numbers.

We are ready to handle negative species.

Definition 28. *The category of $\mathbb{B}\text{-}\mathbb{Z}_2\text{-set}$ species is the category $\mathbb{Z}_2\text{-set}^{\mathbb{B}}$. An object $F : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-set}$ is called a $\mathbb{B}\text{-}\mathbb{Z}_2\text{-set}$ species or an integral species. The category of integral species in n -variables is $\mathbb{Z}_2\text{-set}^{\mathbb{B}^n}$.*

We define operations on integral species using the same formulae as in Definition 9 but instead of using the monoidal structures of gpd we use the corresponding structures in $\mathbb{Z}_2\text{-set}$. The negative functor $N : \mathbb{Z}_2\text{-set}^{\mathbb{B}^n} \rightarrow \mathbb{Z}_2\text{-set}^{\mathbb{B}^n}$ is defined as $N(F)(x, f) = N(F(x, f))$ for F in $\mathbb{Z}_2\text{-set}^{\mathbb{B}^n}$. The valuation map $|\cdot| : \text{Ob}(\mathbb{Z}_2\text{-set}^{\mathbb{B}^n}) \rightarrow \mathbb{Z}[[x_1, \dots, x_n]]$ is given by $|F| = \sum_{a \in \mathbb{N}^n} |F([a])| \frac{x^a}{a!}$.

Theorem 29. $(\mathbb{Z}_2\text{-set}^{\mathbb{B}^n}, N, | \cdot |)$ is a surjective categorification of $\mathbb{Z}[[x_1, \dots, x_n]]$.

We proceed to study the categorification of the ring of rational numbers. Consider the category $\mathbb{Z}_2\text{-gpd} = \text{gpd} \times \text{gpd}$ whose objects are pairs of finite groupoids. Morphisms are given by

$$\mathbb{Z}_2\text{-gpd}((a_1, b_1), (a_2, b_2)) = \text{gpd}(a_1, a_2) \times \text{gpd}(b_1, b_2),$$

for $(a_1, b_1), (a_2, b_2)$ objects of $\mathbb{Z}_2\text{-gpd}$.

Disjoint union $\sqcup : \mathbb{Z}_2\text{-gpd} \times \mathbb{Z}_2\text{-gpd} \rightarrow \mathbb{Z}_2\text{-gpd}$ is given by $(a_1, b_1) \sqcup (a_2, b_2) = (a_1 \sqcup a_2, b_1 \sqcup b_2)$. Cartesian product is given by $(a_1, b_1) \times (a_2, b_2) = (a_1 \times a_2 \sqcup b_1 \times b_2, a_1 \times b_2 \sqcup a_2 \times b_1)$. Distinguished objects are $\emptyset = (\emptyset, \emptyset)$ and $1 = (1, \emptyset)$. The valuation map $| \cdot | : \mathbb{Z}_2\text{-gpd} \rightarrow \mathbb{Z}$ is given by $|(a, b)| = |a| - |b|$, and the negative functor $N : \mathbb{Z}_2\text{-gpd} \rightarrow \mathbb{Z}_2\text{-gpd}$ is given by $N(a, b) = (b, a)$, for (a, b) in $\mathbb{Z}_2\text{-gpd}$. We have the following result.

Theorem 30. $(\mathbb{Z}_2\text{-gpd}, N, | \cdot |)$ is a surjective categorification of \mathbb{Q} .

Objects of $\mathbb{Z}_2\text{-gpd}$ are called \mathbb{Z}_2 -graded groupoids or \mathbb{Z}_2 -groupoids. We write $a - b$ instead of (a, b) for (a, b) in $\mathbb{Z}_2\text{-gpd}$. For example, $-a$ denotes the \mathbb{Z}_2 -groupoid (\emptyset, a) . We are ready to give a full definition of the category of rational species.

Definition 31. The category of $\mathbb{B}\text{-}\mathbb{Z}_2\text{-gpd}$ species is $\mathbb{Z}_2\text{-gpd}^{\mathbb{B}}$. An object $F : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ is called a $\mathbb{B}\text{-}\mathbb{Z}_2\text{-gpd}$ species or a rational species. The category of rational species in n -variables is $\mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n}$.

Operations on rational species are constructed as in Definition 9 but instead of using the monoidal structures of gpd we use the corresponding structures in $\mathbb{Z}_2\text{-gpd}$. The negative functor $N : \mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n} \rightarrow \mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n}$ is given by $N(F)(x, f) = N(F(x, f))$ for F in $\mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n}$. The valuation map $| \cdot | : \text{Ob}(\mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n}) \rightarrow \mathbb{Q}[[x_1, \dots, x_n]]$ is given by $|F| = \sum_{a \in \mathbb{N}^n} |F([a])| \frac{x^a}{a!}$.

Theorem 32. $(\mathbb{Z}_2\text{-gpd}^{\mathbb{B}^n}, N, | \cdot |)$ is a surjective categorification of $\mathbb{Q}[[x_1, \dots, x_n]]$.

Let us introduce examples of negative rational species. The integral sine $\text{Si} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ sends a finite set x to the \mathbb{Z}_2 -groupoid $(-1)^{|x|}\text{Si}(x)$ where $\text{Si}(x)$ is given by

$$\text{Ob}(\text{Si}(x)) = \begin{cases} \{x\} & \text{if } |x| \text{ is odd,} \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\text{Si}(x)(x, x) = \mathbb{Z}_{|x|} \text{ for } |x| \text{ odd.}$$

Proposition 33.

$$|\text{Si}| = \int \frac{\sin(x)}{x} dx.$$

Proof.

$$|\text{Si}| = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \int \frac{\sin(x)}{x} dx.$$

□

Recall that any set x may be regarded as the groupoid whose set of objects is x and with identity morphisms only.

Definition 34. Let G be a groupoid and n a positive integer. The increasing factorial groupoid $G^{(n)}$ is given by

$$G^{(n)} = G \times (G \sqcup [1]) \times \cdots \times (G \sqcup [n-1]).$$

The set of objects of $G^{(n)}$ is

$$\bigsqcup_{I \subseteq [n-1]} \text{Ob}(G) \times \left(\prod_{i \in I} \text{Ob}(G) \right) \times \left(\prod_{i \in [n-1]-I} [i] \right).$$

Thus objects in $G^{(n)}$ are pairs (I, f) such that $I \subseteq [n-1]$ and f is a map with domain $\{0, 1, 2, \dots, n-1\}$ such that $f(0) \in \text{Ob}(G)$, $f(i) \in \text{Ob}(G)$ if $i \in I$, and $f(i) \in [i]$ if $i \notin I$. Morphisms in $G^{(n)}$ are given by

$$G^{(n)}((I, f); (J, g)) = \begin{cases} \emptyset & I \neq J \text{ or } f|_{I^c} \neq g|_{J^c}, \\ G(f(0), g(0)) \times \prod_{i \in I} G(f(i), g(i)) & \text{otherwise.} \end{cases}$$

Proposition 35. $|G^{(n)}| = |G|^{(n)}$ for any finite groupoid G .

Proof.

$$|G^{(n)}| = \left| \prod_{i=0}^{n-1} (G \sqcup [i]) \right| = \prod_{i=0}^{n-1} |G \sqcup [i]| = \prod_{i=0}^{n-1} (|G| + i) = |G|^{(n)}$$

□

Definition 36. Let G be a groupoid such that $|G| = \frac{a}{b} \in \mathbb{Q}_{\geq 0}$. Let $\frac{1}{(1+X)^{\frac{a}{b}}} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ be given on a finite set x by

$$\frac{1}{(1+X)^{\frac{a}{b}}}(x) = (-1)^{|x|} G^{(|x|)}.$$

Theorem 37. For $\frac{a}{b} \in \mathbb{Q}_{\geq 0}$, we have $\left| \frac{1}{(1+X)^{\frac{a}{b}}} \right| = \frac{1}{(1+x)^{\frac{a}{b}}}$.

Proof.

$$\left| \frac{1}{(1+X)^{\frac{a}{b}}} \right| = \sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{b} \right)^{(n)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(-\frac{a}{b} \right)_{(n)} \frac{x^n}{n!} = \frac{1}{(1+x)^{\frac{a}{b}}}.$$

□

Corollary 38. Let F be in $\mathbb{Z}_2\text{-gpd}_0^{\mathbb{B}^n}$. The species $\frac{1}{(1+F)^{\frac{a}{b}}} : \mathbb{B}^n \rightarrow \mathbb{Z}_2\text{-gpd}$ given on a finite set x by

$$\frac{1}{(1+F)^{\frac{a}{b}}}(x) = \bigsqcup_{\pi \in \Pi[x]} (-1)^{|\pi|} G^{(|\pi|)} \times \prod_{b \in \pi} F(b),$$

is such that

$$\left| \frac{1}{(1+F)^{\frac{a}{b}}} \right| = \frac{1}{(1+|F|)^{\frac{a}{b}}}.$$

Proof. Follows from Theorem 10. □

We consider the multiplicative inverse of the valuation of a species.

Theorem 39. *Let F be in $\mathbb{Z}_2\text{-gpd}_0^{\mathbb{B}^n}$. The species $(1 + F)^{-1} : \mathbb{B}^n \rightarrow \mathbb{Z}_2\text{-gpd}$ given by*

$$(1 + F)^{-1}(x, f) = \bigsqcup_{x_1 \sqcup \dots \sqcup x_n = x} (-1)^n F(x_1, f|_{x_1}) \times \dots \times F(x_n, f|_{x_n}),$$

is such that

$$|(1 + F)^{-1}| = \frac{1}{1 + |F|}.$$

Proof. From the identities

$$(1 + F)^{-1}(x, f) = \bigsqcup_{x_1 \sqcup \dots \sqcup x_n = x} (-1)^n F(x_1, f|_{x_1}) \times \dots \times F(x_n, f|_{x_n}) = \left(\sum_{n=0}^{\infty} (-1)^n F^n \right)(x, f),$$

we conclude that $|(1 + F)^{-1}| = \sum_{n=0}^{\infty} (-1)^n |F|^n = \frac{1}{1 + |F|}$. □

Theorem 40. *Let F be in $\mathbb{Z}_2\text{-gpd}_0^{\mathbb{B}^n}$. The rational species $\frac{\mathbb{Z}_a^{\sqcup b}}{1 + \mathbb{Z}_a^{\sqcup b} F} : \mathbb{B}^n \rightarrow \mathbb{Z}_2\text{-gpd}$ is such that*

$$\left| \frac{\mathbb{Z}_a^{\sqcup b}}{1 + \mathbb{Z}_a^{\sqcup b} F} \right| = \frac{1}{\frac{a}{b} + |F|}.$$

Proof.

$$\left| \frac{\mathbb{Z}_a^{\sqcup b}}{1 + \mathbb{Z}_a^{\sqcup b} F} \right| = \frac{b}{a} \frac{1}{\left(1 + \frac{b}{a} |F|\right)} = \frac{1}{\frac{a}{b} + |F|}.$$

□

It is worth while to pay attention to what Theorem 39 and Theorem 40 say and what they do not say. Suppose that we have a species F such that $F(\emptyset) = \emptyset$, then Theorem 39 constructs in a *functorial* way a species $(1 + F)^{-1}$ such that

$$|(1 + F)^{-1}| = \frac{1}{1 + |F|}.$$

Suppose now that we have a rational species such that $F(\emptyset) = G$ is not the empty groupoid. Then $F = G + F_+$ where $F_+(x) = F(x)$ for x a nonempty finite set and $F_+(\emptyset) = \emptyset$. Assume that $|G| = \frac{a}{b}$, then according to Theorem 40 we have that

$$\left| \frac{\mathbb{Z}_a^{\sqcup b}}{1 + \mathbb{Z}_b^{\sqcup a} F_+} \right| |F| = 1.$$

The species $\frac{\mathbb{Z}_a^{\sqcup b}}{1 + \mathbb{Z}_a^{\sqcup b} F_+}$ is constructed in a canonical yet not functorial way from F . Notice the groupoid $\mathbb{Z}_a^{\sqcup b}$ is such that $|\mathbb{Z}_a^{\sqcup b}| = |G|^{-1}$, however the map sending G into $\mathbb{Z}_a^{\sqcup b}$ is not functorial.

5 Bernoulli numbers and polynomials

In this section we use Theorem 39 and Theorem 40 to provide a combinatorial interpretation for Bernoulli numbers and polynomials. Let us first introduce a generalization of the Bernoulli numbers. For N a positive integer, we let the N -projection map $\pi_N : \mathbb{Q}[[x]] \rightarrow \mathbb{Q}[[x]]/(x^N)$ be

$$\text{given by } \pi_N \left(\sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \right) = \sum_{n=0}^{N-1} f_n \frac{x^n}{n!}.$$

Definition 41. Let $f = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$ and $N \in \mathbb{N}_+$ be such that $f_N \neq 0$. The sequence $\{B_{N,n}^f\}_{n=0}^{\infty}$ is called the (f, N) -Bernoulli numbers sequence. It is such that

$$\frac{x^N/N!}{f(x) - \pi_N(f)(x)} = \sum_{n=0}^{\infty} B_{N,n}^f \frac{x^n}{n!}.$$

If $N = 1$ and $f(x) = e^x$, the sequence $B_{1,n}^{e^x}$ is the Bernoulli numbers sequence B_n such that

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

If $N = 2$ and $f(x) = e^x$ we obtain the $B_{2,n}$ Bernoulli numbers studied in [6]. They satisfy

$$\frac{x^2/2!}{e^x - 1 - x} = \sum_{n=0}^{\infty} B_{2,n} \frac{x^n}{n!}.$$

From Example 13 we obtain the following result.

Proposition 42.

$$|\partial\mathbb{Z}| = \frac{e^x - 1}{x}.$$

Theorem 39 implies the following result.

Theorem 43. The rational species $\frac{1}{\partial\mathbb{Z}} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ is such that

$$\left| \frac{1}{\partial\mathbb{Z}} \right| = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

We write $\partial\mathbb{Z} = 1 + \partial\mathbb{Z}_+$, where $\partial\mathbb{Z}_+(x) = \partial\mathbb{Z}(x)$ if $x \neq \emptyset$ and $\partial\mathbb{Z}_+(\emptyset) = \emptyset$. We get

$$\begin{aligned} \frac{1}{\partial\mathbb{Z}}(x) &= \bigsqcup_{x_1 \sqcup \dots \sqcup x_k = x} (-1)^k \partial\mathbb{Z}_+(x_1) \times \dots \times \partial\mathbb{Z}_+(x_k) \\ &= \bigsqcup_{x_1 \sqcup \dots \sqcup x_k = x} (-1)^k \mathbb{Z}(x_1 \sqcup *_1) \times \dots \times \mathbb{Z}(x_k \sqcup *_k). \end{aligned}$$

This implies that

$$\frac{1}{\partial\mathbb{Z}}(x) = \bigsqcup_{x_1 \sqcup \dots \sqcup x_k = x} (-1)^k \mathbb{Z}_{|x_1|+1} \times \dots \times \mathbb{Z}_{|x_k|+1}.$$

Next result gives a combinatorial interpretation of Bernoulli numbers in terms of the cardinality of \mathbb{Z}_2 -graded groupoids.

Corollary 44.

$$B_n = \left| \bigsqcup_{x_1 \sqcup \dots \sqcup x_k = x} (-1)^k \mathbb{Z}_{|x_1|+1} \times \dots \times \mathbb{Z}_{|x_k|+1} \right|.$$

Corollary 45.

$$B_n = \sum_{a_1 + \dots + a_k = n} \frac{(-1)^k n!}{(a_1 + 1)! \dots (a_k + 1)!}.$$

Proof. By Corollary 44 we have

$$B_n = \sum_{a_1 + \dots + a_k = n} \frac{(-1)^k}{(a_1 + 1)! \dots (a_k + 1)!} \binom{n}{a_1, \dots, a_k}.$$

□

The decreasing factorial rational species $\mathbb{Z}_{(N)} : \mathbb{B} \rightarrow \text{gpd}$ is such that for each finite set x , $\text{Ob}(\mathbb{Z}_{(N)}(x)) = \{x\}$ if $|x| \geq N$ and empty otherwise. For $|x| \geq N$ we have

$$\mathbb{Z}_{(N)}(x, x) = \mathbb{Z}_{|x|} \times \mathbb{Z}_{|x|-1} \times \dots \times \mathbb{Z}_{|x|-N+1}.$$

Proposition 46.

$$|1 + N! \partial^N (\mathbb{Z}_{(N)})_+| = N! \frac{e^x - \pi_N(e^x)}{x^N}.$$

Proof. Since

$$|\mathbb{Z}_{(N)}| = \sum_{n=N}^{\infty} \frac{1}{(n)_N} \frac{x^n}{n!},$$

we conclude

$$|1 + N! \partial^N (\mathbb{Z}_{(N)})_+| = N! \sum_{n=N}^{\infty} \frac{x^{n-N}}{n!} = N! \frac{e^x - \pi_N(e^x)}{x^N}.$$

□

More generally we have the following result.

Proposition 47. *Let $F : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ be a rational species such that $F([N]) = 1$. Then*

$$|1 + N! \partial^N (F \times \mathbb{Z}_{(N)})_+| = N! \frac{|F|(x) - \pi_N(|F|)(x)}{x^N}.$$

Proof. Since

$$|F \times \mathbb{Z}_{(N)}| = \sum_{n=N}^{\infty} \frac{|F([n])|}{(n)_N} \frac{x^n}{n!},$$

then

$$|1 + N! \partial^N (F \times \mathbb{Z}_{(N)})_+| = N! \sum_{n=N}^{\infty} \frac{|F([n])|}{n!} \frac{x^{n-N}}{n!} = N! \frac{|F|(x) - \pi_N(|F|)(x)}{x^N}.$$

□

Recall that any group G may be regarded as the groupoid \overline{G} with object 1 and $\overline{G}(1, 1) = G$.

Theorem 48. *Let $F : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ be a rational species such that $F([n]) = 1$. The valuation of the species $\left(1 + N!\partial^N (F \times \mathbb{Z}_{(N)})_+\right)^{-1} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ is the generating series of the $(|F|, N)$ Bernoulli numbers. That is*

$$\left| \left(1 + N!\partial^N (F \times \mathbb{Z}_{(N)})_+\right)^{-1} \right| = \sum_{n=0}^{\infty} B_{N,n}^{[F]} \frac{x^n}{n!}$$

Let us try to digest the meaning of Theorem 48. For a finite set x we have

$$\left(1 + N!\partial^N (F \times \mathbb{Z}_{(N)})_+\right)^{-1}(x) = \bigsqcup_{\sqcup_{i=1}^k x_i = x} (-1)^k \prod_{i=1}^k N!\partial^N (F \times \mathbb{Z}_{(N)})_+(x_i).$$

Thus we get

$$\left(1 + N!\partial^N (F \times \mathbb{Z}_{(N)})_+\right)^{-1}(x) = \bigsqcup_{\sqcup_{i=1}^k x_i = x} (-N!)^k \prod_{i=1}^k F(x_i \sqcup [N]) \times \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [N]).$$

Let us now introduced a generalization of Bernoulli polynomials.

Definition 49. *Let $f \in \mathbb{Q}[[x]]$ and $N \in \mathbb{N}_+$ be such that $f_N \neq 0$. The sequence $B_{N,n}^f(x)$ is called the (f, N) Bernoulli polynomials sequence and is such that*

$$\sum_{n=0}^{\infty} B_{N,n}^f(x) \frac{y^n}{n!} = \frac{(y^N/N!) f(xy)}{f(y) - \pi_N(f)(y)}.$$

For $N = 1$ and $f(y) = e^y$, we obtain the Bernoulli polynomials $B_n(x)$ given by

$$\sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!} = \frac{ye^{xy}}{e^y - 1}.$$

For $N = 2$ and $f(y) = e^y$, we obtain the $B_{2,n}(x)$ polynomials given by

$$\sum_{n=0}^{\infty} B_{2,n}(x) \frac{y^n}{n!} = \frac{(y^2/2!) e^{xy}}{e^y - 1 - y}.$$

For finite sets a and b we use the notation $\text{Bij}(a, b) = \{f : a \rightarrow b \mid f \text{ is bijective}\}$. Let $XZ : \mathbb{B}^2 \rightarrow \text{gpd}$ be the species such that for each pair of finite sets (a, b) one has

$$XY(a, b) = \begin{cases} 1 & \text{if } |a| = |b| = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

The valuation of XY is given by $|XY| = xy \in \mathbb{Q}[[x, y]]$.

Proposition 50. *Let $F \in \mathbb{Z}_2\text{-gpd}^{\mathbb{B}^2}$. The rational species $F(XY)$ is such that $F(XY)(a, b) = F(a)\text{Bij}(a, b)$ for each pair of finite sets a and b . Moreover $|F(XZ)| = |F|(xz)$.*

Proof.

$$F(\text{XY})(a, b) = \bigsqcup_{\pi \in \Pi[x]} F(\pi) \times \prod_{b \in \pi} \text{XY}(a, b) = \bigsqcup_{\substack{f: a \rightarrow b \\ \text{bijective}}} F(a) = F(a) \times \text{Bij}(a, b).$$

The identity $|F(\text{XY})| = |F|(xy)$ follows from Theorem 10. \square

Our next result provides combinatorial interpretation for Bernoulli polynomials.

Theorem 51. *The rational species $\frac{F(\text{XY})}{1 + N! \partial^N (F \times \mathbb{Z}_{(N)})_+} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ is such that*

$$\left| \frac{F(\text{XY})}{1 + N! \partial^N (F \times \mathbb{Z}_{(N)})_+} \right| = \sum_{n=0}^{\infty} B_{N,n}^f(x) \frac{y^n}{n!}.$$

From Definition 9, Theorem 48 and Proposition 50 we see that for any pair of finite sets (a, b) the groupoid

$$\frac{F(\text{XY})}{1 + N! \partial^N (F \times \mathbb{Z}_{(N)})_+}(a, b)$$

is given by

$$\bigsqcup \left((-N!)^k F(c) \times \prod_{i=1}^k F(x_i \sqcup [N]) \times \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [N]) \right)$$

where the disjoint union above runs over subsets all $c \subseteq a$, all injective maps $i : c \rightarrow b$, and all ordered partitions $x_1 \sqcup \dots \sqcup x_k = b \setminus i(c)$.

6 Euler numbers and polynomials

In this section we provide a combinatorial interpretation for Euler numbers and polynomials. It would be interesting to extend our results to the q -Euler numbers and polynomials discussed in [9]. The Euler numbers are denoted by E_n and satisfy

$$\frac{2}{1 + e^x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

The following result should be clear.

Theorem 52. *The rational species $\frac{1}{1 + \mathbb{Z}_2 \text{Exp}_+} : \mathbb{B} \rightarrow \mathbb{Z}_2\text{-gpd}$ is such that*

$$\left| \frac{1}{1 + \mathbb{Z}_2 \text{Exp}_+} \right| = \frac{2}{1 + e^x}.$$

Explicitly for any finite set x we have

$$\frac{1}{1 + \mathbb{Z}_2 \text{Exp}_+}(x) = \sum_{x_1 \sqcup \dots \sqcup x_k = x} (-1)^k (\overline{\mathbb{Z}_2})^k.$$

Next we provide a combinatorial interpretation for Euler polynomials.

Definition 53. *The Euler polynomials sequence is given by*

$$\sum_{n=0}^{\infty} E_n(x) \frac{y^n}{n!} = \frac{2e^{xy}}{1+e^y}.$$

We state our final result.

Theorem 54.

$$\left| \frac{\text{Exp}(XY)}{1 + \overline{\mathbb{Z}}_2 \text{Exp}_+(Y)} \right| = \sum_{n=0}^{\infty} E_n(x) \frac{y^n}{n!}.$$

Explicitly, for any pair of finite sets (a, b) we have

$$\frac{\text{Exp}(XY)}{1 + \overline{\mathbb{Z}}_2 \text{Exp}_+(Y)}(a, b) = \bigsqcup (-1)^k (\overline{\mathbb{Z}}_2)^k,$$

where the disjoint union runs over all subsets $c \subseteq a$, all injective maps $i : c \rightarrow b$, and all ordered partitions $x_1 \sqcup \dots \sqcup x_k = b \setminus i(c)$.

It would be interesting to find combinatorial interpretation for other known sequences of rational numbers. A step forward in that direction has been taken in [4], where a combinatorial interpretation for hypergeometric functions is provided.

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